ISRAEL JOURNAL OF MATHEMATICS 124 (2001), 177–184

RELATIONS AMONG SUMS OF RECIPROCAL POWERS

BY

J. M. AMIGÓ*

Operations Research Center, Miguel Hernández University
03202 Elche (Alicante), Spain
e-mail: jm.amigo@umh.es

ABSTRACT

Some formulas relating different classical sums of reciprocal powers are derived. These relations can be written in a very compact way by means of certain numbers which include Catalan's constant and satisfy simple summation formulas.

1. Introduction

The subject of this paper is the study of the classical numbers

$$\lambda(n) = \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^n} \ (n \geq 2), \quad L(n) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)^n} \ (n \geq 1).$$

As it is well-known, the numbers $\lambda(2n)$ and $L(2n+1)$ can be evaluated in closed form as follows [1, Ch. 23]:

$$\lambda(2n) = (-1)^{n+1} \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!} B_{2n} = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!} |B_{2n}| \ (n \geq 1)$$

where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$ are the Bernoulli numbers and

$$L(2n+1) = (-1)^n \frac{(\pi/2)^{2n+1}}{2(2n)!} E_{2n} = \frac{(\pi/2)^{2n+1}}{2(2n)!} |E_{2n}| \ (n \geq 0)$$

where $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$ are the Euler numbers.

---

* This paper has been partially supported by a DGESIC grant PB97-0342.
Received March 6, 2000 and in revised form July 6, 2000

For $k \geq 2$ define

$$C(k) = \left(\frac{\pi}{2}\right)^{k-1} \sum_{j=0}^{\infty} \frac{L(2j+1)}{(2j+1)\cdots(2j+k)} = \sum_{j=0}^{\infty} \frac{(\pi/2)^{k+2j}}{(k+2j)!} |E_{2j}|. \tag{1.1}$$

It will be shown below that these numbers can be alternatively expressed as

$$C(k) = \frac{1}{2(k-1)!} \int_0^{\pi/2} \frac{t^{k-1}}{\sin t} dt \tag{1.2}$$

and include Catalan's constant $\mathbf{G} = L(2) = 0.91596\ldots$ for $k = 2$. Besides being interesting by themselves, these "Catalan's constants" $C(k)$ have the property of relating the values $\lambda(2n+1)$ and $L(2n)$ to the elementary values $\lambda(2n)$ and $L(2n+1)$ as, for example, in

$$\sum_{j=0}^{n-1} (-1)^j \frac{(\pi/2)^{2j+1}}{(2j+1)!} L(2n-2j) = \frac{4}{\pi} \sum_{j=0}^{n-1} (-1)^j C(2j+2) \lambda(2n-2j). \tag{1.3}$$

This and other formulas expressing the numbers $C(k)$ by means of the numbers $\lambda(n)$ and $L(n)$ and, conversely, $\lambda(n)$ and $L(n)$ via $C(k)$ (see Proposition 2.2 and 2.4, respectively) will be proved in the next section. In the last section, some summation formulas for the numbers $C(k)$ are derived.

2. Statements and proofs

Define the generating functions $\Lambda(x)$ and $\mathcal{L}(x)$ by

$$\Lambda(x) = \sum_{n=2}^{\infty} \lambda(n) x^{n-1}, \quad \mathcal{L}(x) = \sum_{n=1}^{\infty} L(n) x^{n-1}$$

(since $\lim_{n\to\infty} \lambda(n) = \lim_{n\to\infty} L(n) = 1$, these formal power series converge only for $|x| < 1$) and denote by $\Lambda_+(x)$ and $\Lambda_-(x)$ the even and odd parts, respectively, of $\Lambda(x)$, and similarly for $\mathcal{L}_\pm(x)$. Then [1, 4.3.67/69]

$$\tan\left(\frac{\pi x}{2}\right) = \frac{4}{\pi} \sum_{n=1}^{\infty} \lambda(2n) x^{2n-1} = \frac{4}{\pi} \Lambda_-(x) \tag{2.1}$$

and

$$\sec\left(\frac{\pi x}{2}\right) = \frac{4}{\pi} \sum_{n=0}^{\infty} L(2n+1) x^{2n} = \frac{4}{\pi} \mathcal{L}_+(x) \tag{2.2}$$

if $|x| < 1$.

Substituting the definitions of $\lambda(n)$ and $L(n)$ into the corresponding generating functions, we find

$$\Lambda(x) = \sum_{\nu=0}^{\infty}\left(\frac{1}{2\nu+1-x} - \frac{1}{2\nu+1}\right), \ \mathcal{L}(x) = \sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{2\nu+1-x}.$$

PROPOSITION 2.1: *The integral representation* (1.2) *holds.*

Proof: Substitute

$$\begin{aligned}\frac{1}{(2j+1)\cdots(2j+k)} =& \frac{(2j)!}{(2j+k)!} = \frac{B(k,2j+1)}{(k-1)!}\\ =& \frac{1}{(k-1)!}\int_0^1 t^{k-1}(1-t)^{2j}dt\end{aligned}$$

in (1.1) to obtain

$$\begin{aligned}C(k) =& \left(\frac{\pi}{2}\right)^{k-1}\frac{1}{(k-1)!}\int_0^1 t^{k-1}\left(\sum_{j=0}^{\infty}L(2j+1)(1-t)^{2j}\right)dt\\ =& \left(\frac{\pi}{2}\right)^{k-1}\frac{1}{(k-1)!}\int_0^1 t^{k-1}\mathcal{L}_{+}(1-t)dt\\ =& \frac{1}{2}\left(\frac{\pi}{2}\right)^{k}\frac{1}{(k-1)!}\int_0^1 t^{k-1}\sec\left(\frac{\pi(1-t)}{2}\right)dt\\ =& \frac{1}{2(k-1)!}\int_0^1\frac{(\pi t/2)^{k-1}}{\sin\frac{\pi t}{2}}d\frac{\pi t}{2} = \frac{1}{2(k-1)!}\int_0^{\pi/2}\frac{t^{k-1}}{\sin t}dt\end{aligned}$$

where (2.2) has been used. ∎

Since $(2/\pi)t \le \sin t \le t$ for $0 \le t \le \pi/2$, we get from (1.2) the bounds

$$\frac{(\pi/2)^{k-1}}{2(k-1)(k-1)!} \le C(k) \le \frac{(\pi/2)^{k}}{2(k-1)(k-1)!}. \tag{2.3}$$

Thus, $\lim_{k\to\infty} C(k) = 0$.

PROPOSITION 2.2: *The numbers $C(k)$ ($k \ge 2$) can be evaluated in terms of $L(2n)$ and $\lambda(2n+1)$ by the formula*

$$C(k) = \sum_{j=1}^{\lfloor k/2\rfloor}(-1)^{j-1}\frac{(\pi/2)^{k-2j}}{(k-2j)!}L(2j) + \begin{cases}0 & (k \text{ even }),\\ (-1)^{(k-1)/2}\lambda(k) & (k \text{ odd }).\end{cases} \tag{2.4}$$

Proof: The imaginary part of the geometric series $\sum_{\nu=0}^{\infty} e^{i\nu t} = (1-e^{it})^{-1} = \frac{1}{2} + \frac{1}{2} i \cot\frac{t}{2}$ $(0 < t < 2\pi)$ reads $\sum_{\nu=1}^{\infty} \sin\nu t = \frac{1}{2}\cot\frac{t}{2}$, hence

$$\begin{aligned}\sum_{\nu=0}^{\infty} \sin(2\nu+1)t =& \sum_{\nu=1}^{\infty} \sin\nu t - \sum_{\nu=1}^{\infty} \sin 2\nu t \\ =& \frac{1}{2}\left(\cot\frac{t}{2} - \cot t\right) = \frac{1}{2}\frac{1}{\sin t}\end{aligned}$$

and

$$\begin{aligned}C(k) =& \frac{1}{2(k-1)!}\int_0^{\pi/2} \frac{t^{k-1}}{\sin t} dt \\ =& \frac{1}{(k-1)!}\sum_{\nu=0}^{\infty}\int_0^{\pi/2} t^{k-1}\sin(2\nu+1)t dt.\end{aligned}$$

Eq. (2.4) follows now from the formula

$$\begin{aligned}&\frac{1}{(k-1)!}\int_0^{\pi/2} t^{k-1}\sin(2\nu+1)t dt \\ &= \sum_{j=1}^{\lfloor k/2\rfloor}(-1)^{j-1}\frac{(\pi/2)^{k-2j}}{(k-2j)!}\frac{(-1)^{\nu}}{(2\nu+1)^{2j}} + \begin{cases} 0 & (k \text{ even }) \\ (-1)^{(k-1)/2}\frac{1}{(2\nu+1)^k} & (k \text{ odd }) \end{cases}\end{aligned}$$

which can be easily proved by integration by parts and induction. ∎

In particular, setting $k=2$ in (1.2), (1.1) and (2.4), we have

$$\frac{1}{2}\int_0^{\pi/2} \frac{t}{\sin t} dt = C(2) = \frac{\pi}{2}\sum_{j=0}^{\infty}\frac{L(2j+1)}{(2j+1)(2j+2)} = L(2).$$

(This integral representation of **G** appears, for ex., in [3, 3.747(2)].) More generally, each $C(k)$ embodies, due to its definition (1.1) and to formula (2.4), a relation among all elementary values $L(2n+1)$ and the values $L(2)$, $L(4), \ldots$, $L(2k)$, supplemented with $\lambda(k)$ if k is odd, namely,

$$\sum_{j=0}^{\infty}\frac{L(2j+1)}{(2j+1)\cdots(2j+2k)} = \sum_{j=1}^{k}(-1)^{j-1}\frac{(2/\pi)^{2j-1}}{(2k-2j)!}L(2j),$$

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{L(2j+1)}{(2j+1)\cdots(2j+2k+1)} =& \sum_{j=1}^{k}(-1)^{j-1}\frac{(2/\pi)^{2j-1}}{(2k-2j+1)!}L(2j) \\ &+ (-1)^k\left(\frac{2}{\pi}\right)^{2k}\lambda(2k+1)\end{aligned}$$

for $k \geq 1$.

Remark 2.1: Eq. (1.2) is formally analogous to [2, p. 35]

$$L(s) = \frac{1}{2\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\cosh t} dt \quad (\mathcal{R}(s) > 1),$$

where $L(s)$ is the analytic continuation of $\sum_{\nu=0}^{\infty}(-1)^s/(2\nu+1)^s$, $\mathcal{R}(s) > 1$, to the complex plane. The equality $L(2) = C(2)$ can be directly checked by integrating the complex function $f(z) = z/\cosh z$ along the (positively oriented) rectangular path with vertices $(0,0)$, $(R,0)$ (R,π) and $(0,\pi)$ (surrounding the pole at $(0,\pi/2)$) and letting $R \to \infty$.

Define next the generating function

$$\mathcal{C}(x) = \sum_{n=2}^{\infty} C(n)(ix)^{n-1}.$$

Then

$$\mathcal{C}(x) = \frac{1}{2}\int_0^{\pi/2} \frac{dt}{\sin t} \sum_{n=2}^{\infty} \frac{(ixt)^{n-1}}{(n-1)!} = \frac{1}{2}\int_0^{\pi/2} \frac{e^{ixt}-1}{\sin t} dt. \tag{2.5}$$

Furthermore, let $\mathcal{C}_+(x)$ and $\mathcal{C}_-(x)$ be the even and odd parts of $\mathcal{C}(x)$, i.e.,

$$\mathcal{C}_+(x) = \sum_{n=1}^{\infty} C(2n+1)(ix)^{2n} = \sum_{n=1}^{\infty}(-1)^n C(2n+1)x^{2n},$$
$$\mathcal{C}_-(x) = \sum_{n=1}^{\infty} C(2n)(ix)^{2n-1} = i\sum_{n=1}^{\infty}(-1)^{n+1} C(2n)x^{2n-1}.$$

Therefore, if x is meant to be real, $\mathcal{C}_+(x)$ and $\frac{1}{i}\mathcal{C}_-(x)$ are the real and imaginary part, respectively, of $\mathcal{C}(x)$.

Remark 2.2: Owing to the bounds (2.3), the power series defining $\mathcal{C}(x)$ converges for all $x \in \mathbb{C}$.

PROPOSITION 2.3: *The following relations hold:*

$$\mathcal{C}_+(x) = \Lambda_+(x) - \sin\left(\frac{\pi x}{2}\right)\mathcal{L}_-(x), \quad \mathcal{C}_-(x) = i\cos\left(\frac{\pi x}{2}\right)\mathcal{L}_-(x) \tag{2.6}$$

so that

$$\mathcal{C}(x) = \Lambda_+(x) + ie^{i\pi x/2}\mathcal{L}_-(x).$$

Proof: Indeed,

$$\begin{aligned}\sin\left(\frac{\pi x}{2}\right)\mathcal{L}_-(x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\pi/2)^{2n-1}}{(2n-1)!}x^{2n-1}\sum_{n=1}^{\infty}L(2n)x^{2n-1}\\ &= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(-1)^{n-j}\frac{(\pi/2)^{2n+1-2j}}{(2n+1-2j)!}L(2j)\right)x^{2n}\\ &= \sum_{n=1}^{\infty}(-1)^{n}\left(\sum_{j=1}^{n}(-1)^{j}\frac{(\pi/2)^{2n+1-2j}}{(2n+1-2j)!}L(2j)\right)x^{2n}\end{aligned}$$

and

$$\begin{aligned}\cos\left(\frac{\pi x}{2}\right)\mathcal{L}_-(x) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{(\pi/2)^{2n}}{(2n)!}x^{2n}\sum_{n=1}^{\infty}L(2n)x^{2n-1}\\ &= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(-1)^{n-j}\frac{(\pi/2)^{2n-2j}}{(2n-2j)!}L(2j)\right)x^{2n-1}\\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\left(\sum_{j=1}^{n}(-1)^{-j-1}\frac{(\pi/2)^{2n-2j}}{(2n-2j)!}L(2j)\right)x^{2n-1}.\end{aligned}$$

The statement follows from (2.4). ∎

The following Proposition is a kind of converse of Proposition 2.2.

PROPOSITION 2.4: *The numbers $L(2n)$ and $\lambda(2n+1)$ can be expressed in terms of the $C(k)$ and the elementary values $L(2n+1)$ and $\lambda(2n)$ by*

$$\begin{aligned}L(2n) &= \frac{4}{\pi}\sum_{j=1}^{n}(-1)^{j-1}L(2n-2j+1)C(2j),\\ \lambda(2n+1) &= \frac{4}{\pi}\sum_{j=1}^{n}(-1)^{j-1}\lambda(2n-2j+2)C(2j)+(-1)^{n}C(2n+1).\end{aligned}$$

Proof: From (2.6) it follows that

$$\mathcal{L}_-(x) = \frac{1}{i}\sec\left(\frac{\pi x}{2}\right)\mathcal{C}_-(x)$$

and

$$\Lambda_+(x) = \mathcal{C}_+(x) + \sin\left(\frac{\pi x}{2}\right)\mathcal{L}_-(x) = \mathcal{C}_+(x) + \frac{1}{i}\tan\left(\frac{\pi x}{2}\right)\mathcal{C}_-(x)$$

where

$$\begin{aligned}\sec\left(\frac{\pi x}{2}\right)\mathcal{C}_-(x) &= \frac{4}{\pi}\sum_{n=0}^{\infty}L(2n+1)x^{2n}\sum_{n=1}^{\infty}C(2n)(ix)^{2n-1}\\ &= i\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(-1)^{j-1}L(2n-2j+1)C(2j)\right)x^{2n-1}\end{aligned}$$

and

$$\begin{aligned}\tan\left(\frac{\pi x}{2}\right)\mathcal{C}_-(x) =& \frac{4}{\pi}\sum_{n=1}^{\infty}\lambda(2n)x^{2n-1}\sum_{n=1}^{\infty}C(2n)(ix)^{2n-1} \\ =& i\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}(-1)^{j-1}\lambda(2n-2j+2)C(2j)\right)x^{2n}\end{aligned}$$

if $|x| < 1$. ∎

Remark 2.3: Eq. (1.3) is nothing else but the formula

$$\sin\left(\frac{\pi x}{2}\right)\mathcal{L}_-(x) = \frac{1}{i}\tan\left(\frac{\pi x}{2}\right)\mathcal{C}_-(x)$$

written explicitly.

3. Summation formulas for the $C(k)$'s

A first (integral) summation formula follows trivially from the very definition of the generating function $\mathcal{C}(x)$ and Eq. (2.5), namely,

$$\sum_{n=2}^{\infty} C(n) = \mathcal{C}(1/i) = \frac{1}{2}\int_0^{\pi/2}\frac{e^t-1}{\sin t}dt.$$

Also from (2.5) we have

$$\begin{aligned}\mathcal{C}(x+1) - \mathcal{C}(x-1) =& i\int_0^{\pi/2} e^{ixt}dt = \frac{1}{x}(e^{i\pi x/2}-1) \\ =& \frac{1}{x}\left(\cos\frac{\pi x}{2} - 1\right) + i\frac{1}{x}\sin\frac{\pi x}{2}\end{aligned}$$

so that, for $x \in \mathbb{R}$, the equations

$$\begin{aligned}\mathcal{C}_+(x+1) - \mathcal{C}_+(x-1) =& \frac{1}{x}\left(\cos\frac{\pi x}{2} - 1\right), \\ \mathcal{C}_-(x+1) - \mathcal{C}_-(x-1) =& i\frac{1}{x}\sin\frac{\pi x}{2}\end{aligned} \tag{3.1}$$

hold. From these formulas, some others can be derived in a variety of ways.

PROPOSITION 3.1: *The following summation formulas hold:*

(1)

$$\sum_{k=1}^{\infty}(-1)^{k+1}2^{2k}C(2k+1) = 1 = \sum_{k=1}^{\infty}(-1)^{k+1}2^{2k-1}C(2k).$$

(2) *For* $n = 0, 1, 2, \ldots$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\binom{n+2k-1}{n}C(n+2k) = \frac{(\pi/2)^{n+1}}{2(n+1)!}.$$

(3)

$$\sum_{k=1}^{\infty}(-1)^{k+1}C(2k) = \frac{\pi}{4}, \quad \sum_{k=1}^{\infty}(-1)^{k+1}kC(2k+1) = \frac{\pi^2}{32}.$$

Proof: (1) Set $x = 1$ in (3.1) to obtain

$$\mathcal{C}_+(2) = -1, \quad \mathcal{C}_-(2) = i.$$

(2) In fact,

$$\begin{aligned}\mathcal{C}_+(x+1) - \mathcal{C}_+(x-1) &= \sum_{k=1}^{\infty}(-1)^k C(2k+1)((x+1)^{2k} - (x-1)^{2k}) \\ &= 2\sum_{k=1}^{\infty}(-1)^k C(2k+1)\sum_{j=0}^{k-1}\binom{2k}{2j+1}x^{2j+1} \\ &= 2\sum_{j=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{j+k}\binom{2j+2k}{2j+1}C(2j+1+2k)\right)x^{2j+1}\end{aligned}$$

and, analogously,

$$\mathcal{C}_-(x+1) - \mathcal{C}_-(x-1) = 2i\sum_{j=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{j+k+1}\binom{2j+2k-1}{2j}C(2j+2k)\right)x^{2j}.$$

Comparison with (3.1) yields the result for $n = 2j + 1$ and $n = 2j$, respectively.

(3) Set $n = 0$ and $n = 1$ in the formula of (2). ∎

Of course, all these summation formulas can be also checked using the integral representation (1.2).

ACKNOWLEDGEMENT: I am very grateful to the referee of this paper whose kind recommendations were decisive for the final presentation of the results.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1972.

[2] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (5th edition), Academic Press, New York, 1994.